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Lec 6:

Nuclear Energy Production in Stars:

Most observed stars live on thermonuclear fusion. In a typical process a number j of lighter nuclei fuse to form a heavier one y. This can only happen if the sum of lighter nuclei is larger than the mass of heavier nucleus:

$$\Delta M = \sum_j M_j - M_y > 0 \quad (\Delta M: \text{mass defect})$$

The energy produced in the fusion process is $E = \Delta M c^2$.

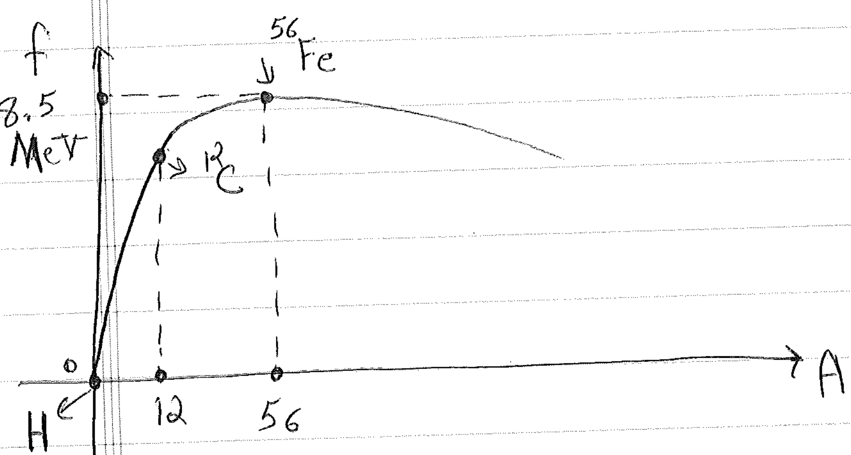
The deficiency of mass is an aspect of the fact that heavie[r] nucleus must be more tightly bound. The binding energy of a nucleus with Z protons and A-Z neutrons (thus atomic ma[ss] number A) is:

$$E_B = \left\{ \left[Z m_p + (A-Z) m_n \right] - M_{nucl} \right\} c^2$$

The binding fraction is defined as the average binding energy per nucleon:

$$f = \frac{E_B}{A}$$

Typical values of f are around 8 MeV (except for Hydrogen where $A=1$). The behavior of f as a function of A for stable nuclei is shown below:



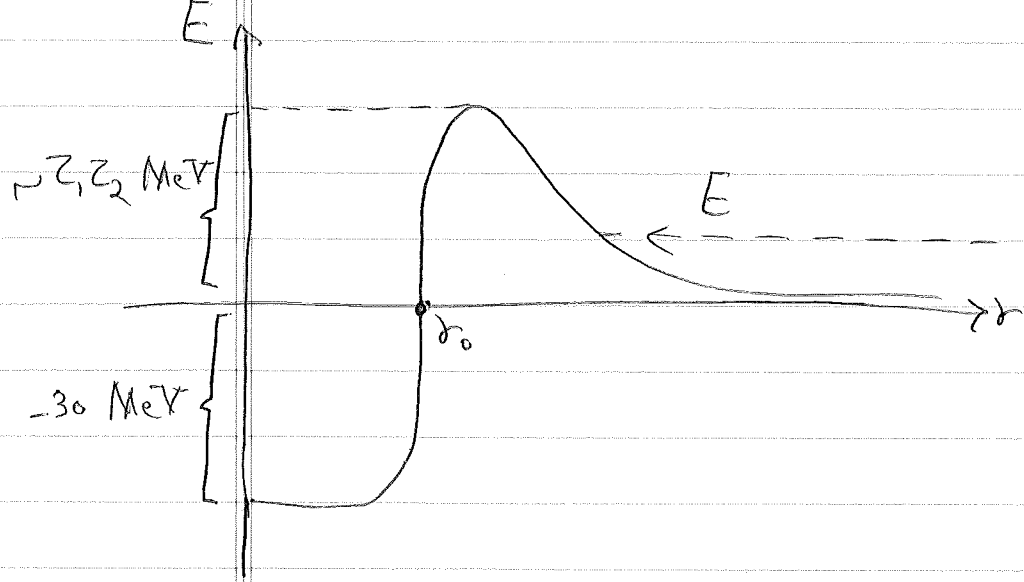
It is seen that f sharply rises as A increases, then flattens out and reaches a maximum at $A=56$ (^{56}Fe), and drops slowly. The rise can be understood as a surface effect. Nucleons inside a nucleus feel the short range nuclear force that is attractive. Those nucleons that are close to the surface of nucleus see a smaller number of nucleons than those that are in the interior. Therefore they feel less attraction.

As A increases, so does the radius of the nucleus. The number of nucleons that are close to the surface goes as $(\text{radius})^2$, while the number of nucleons in the interior is proportional to $(\text{radius})^3$. As a result, the contributions of nucleons that are close to the surface to the binding energy decreases as $(\text{radius})^{-1}$ by increasing A. The net effect is having a larger binding fraction f for larger values of A.

For $A > 56$ things turn around. This can be understood from the fact that larger nuclei have a larger number of protons. Since neutrons are fermions, Pauli exclusion principle implies that larger A also requires larger Z as one cannot feel the energy eigenstates with neutrons only (unless going to higher energy states). Protons feel long range Coulomb repulsion, which decreases the binding fraction. This is why f decreases beyond ^{56}Fe.

We see that ^{56}Fe is the most tightly bound stable nucleus. As mentioned before, it is the endpoint of nuclear burning inside (massive) stars.

In order to obtain a fusion of charged particles, they have to be brought so close that they feel the short range strong nuclear force. The problem will be that of a particle in a potential consisting of the short range nuclear force and long range Coulomb force:



Note that the short range force results in a potential well (because of its attractive nature) that is ~ 30 MeV deep. The Coulomb force results in a potential barrier (due to its repulsive nature) that has a height $\sim Z_1 Z_2$ MeV, with Z_1, Z_2 being the number of protons in the two nuclei. The range of nuclear force follows:

$r_0 \approx A^{1/3}\ 1.44 \times 10^{-13}$ cm

This is ~ 1000 times smaller than the size of Hydrogen atom.

Lets Consider two protons ($Z_1 = Z_2 = 1$). The Coulomb barrier in this case is ~ 1 MeV. The typical energy E at the center of Sun is the average thermal energy, which for a temperature $T \simeq 10^7$ K is ~ 1 keV. It seems that there is no way to overcome the Coulomb barrier. One should remember though that at a given temperature T a particle can have any energy E according to Maxwell-Boltzmann distribution

$f(E) \propto \exp\left(\frac{-E}{k_B T}\right)$. The probability to have an energy ~ 1 MeV at the temperature $T \simeq 10^7$ K will therefore be $\sim \exp\left(-\frac{1\text{ MeV}}{1\text{ keV}}\right) \sim \exp(-1000)$. This is 10^{-434} !! It is extremely small: the total number of protons inside Sun is $\sim 10^{57}$ (the number of nucleons in the observable part of the universe is $\sim 10^{88}$). This implies that it will not be possible to even have one fusion process to occur within the current age of the universe.

Our argument has been at the classical level so far. A particle approaching the potential barrier from large distances can actually penetrate the barrier in quantum mechanics. The tunneling probability is given by:

$$\boxed{P_0 = P_0\, E^{-\frac{1}{2}}\, e^{-2\pi\eta} \quad , \quad \eta = \left(\frac{m}{2}\right)^{\frac{1}{2}} \frac{Z_1 Z_2 e^2}{\hbar E^{\frac{1}{2}}}} \quad *$$

Here m is the reduced mass of the system ($m = \frac{m_p}{2}$ for two protons). For $T \simeq 10^{7}$ K we find $P_0 \sim 10^{-20}$. The probability is very small, but not too small. Considering the number of protons inside Sun ($\sim 10^{57}$), a large number of protons can still participate in fusion. The smallness of probability implies that Hydrogen burning is slow and lasts long. This is the reason why the main sequence takes so long ($\sim 10^{10}$ years for Sun).

We note that P_0 crucially depends on η. We see from equation * that η is proportional to $Z_1 Z_2$. The probability for fusion of two α particles ($Z_1 = Z_2 = 2$) at the same temperature as inside Sun will therefore be extremely small ($P_0 \sim 10^{-80}$). Helium fusion can only occur at large temperature ($P_0 \sim 10^{-20}$ for $T \simeq 10^{8}$ K). This is the reason why Helium burning does not start until further contraction of the core (at the end of Hydrogen burning).

Nuclear Cross-sections:

Consider a reaction of the nucleus X with the particle a by which the nucleus Y and the particle b are formed:

$$a + X \rightarrow Y + b$$

This process is represented by $X(a, b)Y$. The reaction probability depends on nuclear details.

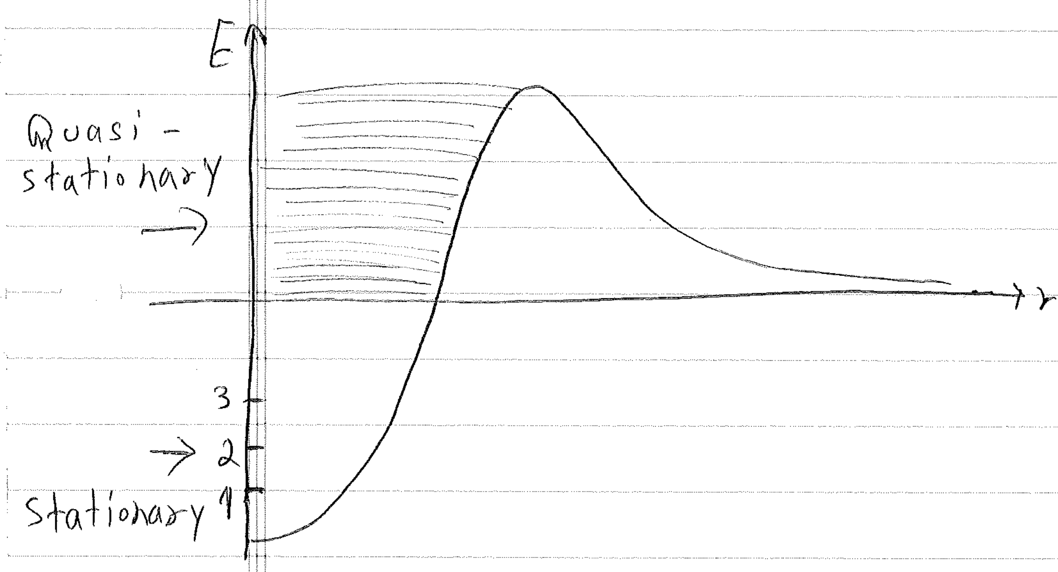
Lets first consider an important case where the reaction can proceed through a resonance. In this case an excited compound nucleus C^* may form containing both X and a. C^* can generally decay to several eigenstates:

$$C^* \rightarrow X + a,\ Y_1 + b_1,\ Y_2 + b_2, \ldots$$

The first is the reproduction of the original particles. In the other channels $b_1, b_2, \ldots$ may be neutrons, protons, α particle, photon, etc. Note that electron emission is negligible as it happens via weak interactions. We also note that C^*

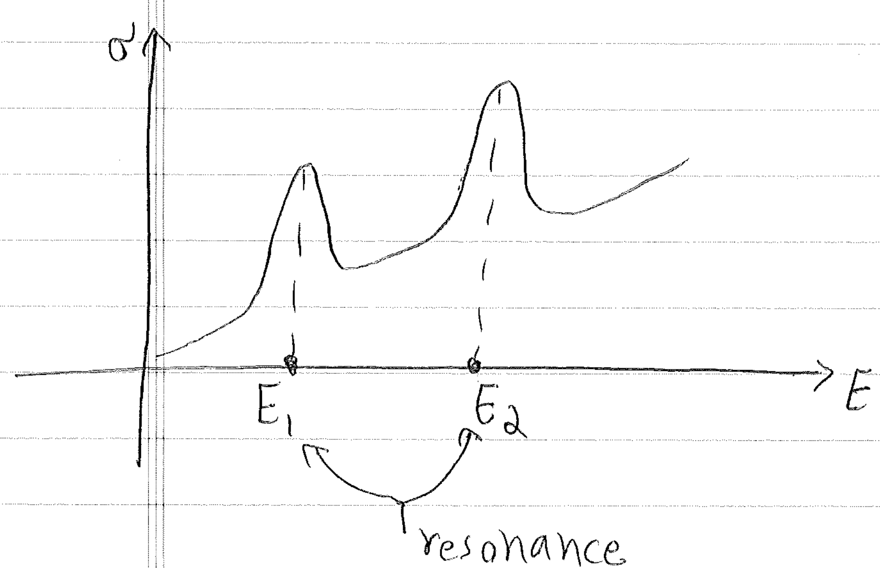
decay via a certain channel requires fulfillment of conservation laws (energy, momentum, angular momentum, symmetries).

It is important to know the energy levels of the compound nucleus C^*. Schematic sketch of energy levels is as follows;



Energy levels with $E<0$ are discrete and correspond to bound states. Excitation of the ground state will result in decay by electromagnetic transition with the emission of γ rays. Their lifetime is typically large.

The compound nucleus can eject a particle via tunneling if it has energy $E>0$. These states are quasi-stationary. If E is such that tunneling probability is high enough, the excited nucleus can decay quickly via particle emission. The colliding particles in this case form a "resonance". At resonance energies E_{res} the probability for a reaction (thus reaction cross-section) is significantly enhanced. The energy dependence of the cross-section can be schematically shown as follows:



The dependence of $\sigma_{(E)}$ on E is given by the relation:

$$\sigma_{(E)} \sim \boxed{\pi \lambda^2_{deB}\, P_0(E)\, \zeta_{(E)}}$$

Here $\pi \lambda^2_{deB}$ represents the geometrical cross-section, where $\lambda_{deB} = \frac{\hbar}{(2mE)^{1/2}}$, see equation *, $P_0(E)$ represents penetration to the Coulomb barrier, and $\zeta(E)$ represents resonance effect. Near a resonance we have:

$$\zeta(E) = \text{Const} \frac{1}{(E-E_{res})^2 + (\frac{\Gamma}{2})^2}$$

Where Γ is the resonance width. Away from a resonance $\zeta \to 1$.

One can write:

$$\boxed{\sigma(E) = S(E)\, E^{-1}\, e^{-2\pi\eta}}$$

The "Astrophysical cross-section factor" S contains all intrinsic properties of the reaction under consideration. It is determined by laboratory measurements.

The difficulty with measurements of $S(E)$ is that they are possible at rather high energies $\gtrsim 0.1$ MeV (because of the small cross-sections). This is a factor of 10 larger than energies relevant for astrophysical applications (recall that $T \sim 1$ keV at the center of Sun). One therefore needs to extrapolate the measured values toward lower energies over a wide range of E. This can be done reliably for non-resonant reactions, in which case S is a slowly varying function of E that can be considered nearly constant. The real problem arises when there are resonances in the range over which the extrapolation is made.

Thermonuclear Reaction Rates:

Lets consider a reaction between particle types j and k. The total number of reactions per unit volume per unit time is:

$$r_{jk} = \frac{n_j n_k}{1+\delta_{jk}} \sigma v$$

Here n_j, n_k are the number density of j and k respectively and v is the relative velocity. δ_{jk} is the Kronecker delta function, which is introduced to avoid double counting in the case of identical particles. The two particles can have any relative velocity v (with corresponding kinetic energy $E = \frac{1}{2} m v^2$, m being the reduced mass) according to the Maxwell-Boltzmann distribution:

$$f_{(E)} = \frac{2}{\sqrt{\pi}} \frac{E^{\frac{1}{2}}}{(k_B T)^{3/2}} e^{-\frac{E}{k_B T}} \quad \text{(non-relativistic regime)}$$

Here we have excluded very high densities where quantum effects become important (like in neutron stars) and one needs to use Fermi-Dirac distributions (for fermions).

The thermally averaged reaction rate is given by:

$$r_{jk} = \frac{n_j n_k}{1 + \delta_{jk}} \langle \sigma v \rangle$$

Where:

$$\langle \sigma v \rangle = \int_0^\infty \sigma_{(E)} v f_{(E)} d_E = \frac{2^{3/2}}{(m\pi)^{1/2}} \frac{1}{(k_B T)^{3/2}} \int_0^\infty S_{(E)} e^{-\frac{E}{k_B T}} e^{-\frac{\bar{\eta}}{E^{\frac{1}{2}}}} d_E$$

Here we have defined:

$$\bar{\eta} = 2\pi\eta E^{\frac{1}{2}} = \pi (2m)^{1/2} \frac{Z_j Z_k e^2}{\hbar}$$

Focusing on non-resonant reactions, we can approximate $S(E) \simeq S_0 =$ Constant. The integrand is then the product of two exponentials. It will have appreciable values around a well-defined maximum, which is called Gamow peak:

$$E_0 = \frac{1}{2}(\bar{\eta} k_B T)^{\frac{2}{3}} = \left[\left(\frac{m}{2}\right)^{1/2} \pi \frac{Z_j Z_k e^2 k_B T}{\hbar}\right]^{2/3}$$

Considering the lowest order expansion of the integrand around Gamow peak, it can be approximated by a Gaussian, which results in:

$$\langle \sigma v \rangle = \frac{4}{3}\left(\frac{2}{m}\right)^{1/2} \frac{1}{(k_B T)^{1/2}} S_0 \tau^{\frac{1}{2}} e^{-\tau} \boxed{\sim \tau^2 e^{-\tau}} \quad **$$

Where:

$$\tau = 3\frac{E_0}{k_B T} = 3\left[\pi \left(\frac{m}{2k_B T}\right)^{1/2} \frac{Z_j Z_k e^2}{\hbar}\right]^{\frac{2}{3}}$$

It is seen that $\tau \sim T^{-\frac{1}{3}}$. From equation ** one finds:

$$\boxed{\nu \equiv \frac{\partial \ln \langle \sigma v \rangle}{\partial \ln T} = \frac{\tau}{3} - \frac{2}{3}}$$

For most reactions $\tau \gg 2$, which implies $\nu \simeq \frac{\tau}{3}$. For reactions between the lightest nuclei $\nu \simeq 5$, and it can easily attain values around (and above) $\nu \simeq 20$. Therefore thermonuclear reaction rates are strongly varying function of T, and this temperature sensitivity has a clear influence on stellar models.

One comment is in order at this point. There is a partial shielding of the Coulomb potential of the nuclei, due to the negative field of neighbouring electrons, which decreases the Coulomb barrier height. This, however, plays a role only at very high densities.

Finally, for resonant reactions the situation depends on the location of the resonance. If the resonance energy is faraway from Gamow peak, resonance may be unimportant. But the integrand may be dominated by a strong peak at the resonance energy.